Commutative Banach Algebras and the Gelfand Representation Theorem

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1 Topological Linear Spaces

Though Banach algebras are themselves normed linear spaces, the Gelfand representation theorem involves a linear space which is not endowed with a norm, so I will first make some comments concerning topological linear spaces.

Definition 1. *Given a topological space (X, τ), a collection $\sigma \subset \tau$ is said to be a* sub-basis *of τ if every element of τ is a union of finite intersections of elements of σ.*

Moreover, if we have an arbitrary collection σ of subsets of X, then the collection of all unions of finite intersections of elements of σ, together with $\emptyset$ and X, is easily seen to form a topology of X (and, of course, σ is a sub-basis for this topology).

Definition 2. *Let X be any set, let Γ be an arbitrary index set and for each $\gamma \in \Gamma$, let f_γ be a mapping from X to a topological space (X_γ, τ_γ). Let $\mathcal{F} := \{f_\gamma : \gamma \in \Gamma\}$. Then the* weak topology generated by $\mathcal{F}$ *is the topology generated by the sub-basis*

$$\sigma = \{f_\gamma^{-1}(U_\gamma) : U_\gamma \in \ \tau_\gamma(\gamma \in \Gamma)\}$$

this topology is denoted by $\sigma(X, \mathcal{F})$.

Remark 3. *The importance of the weak topology is that it is the weakest topology for which each of the functions f_γ is continuous from $(X, \sigma(X, \mathcal{F}))$ to (X_γ, τ_γ).*

Definition 4. *Let X be a normed linear space, X^* its dual, X^{**} its second dual and for each $x \in X$ let $\hat{x}$ denote the corresponding element of X^{**} as described in lectures, then the* weak-star topology $\sigma(X^*, X)$ *on X^* is the topology generated by the elements $\hat{x}$.*

Without proof, I will use the following results for a normed linear space X:

Theorem 5. *The weak-star topology $\sigma(X^*, X)$ is Hausdorff.*

Theorem 6 (Alaoglu)**.** *The unit ball $B(X^*) := \{x \in X : \|x\| \leq 1\}$ is compact in the weak-star topology.*

2 Banach Algebras

Banach algebras are a kind of Banach space which generalise the spaces of bounded linear operators. Indeed, we have the following definition:

Definition 7. *Let A be a complex Banach space. A is said to be a* Banach algebra *if there is a multiplication defined on A such that $\forall \lambda \in \mathbb{C}$ and $\forall x, y, z \in A$,*

1. $x(yz) = (xy)z$;
2. $x(y+z) = xy + xz$ *and* $(x+y)z = xz + yz$;
3. $\lambda(xy) = (\lambda x)y = x(\lambda y)$;
4. $\|xy\| \leq \|x\| \, \|y\|$;

Moreover, the Banach algebra is said to have a unit *if $\exists e \in A$ such that $\forall x \in A$, $ex = xe = x$ and $\|e\| = 1$. The algebra is said to be* commutative *if for any $x, y \in A$, $xy = yx$. I shall refer to commutative Banach algebras as CB algebras.*

Note an equivalent defintion of a Banach algebra is that it is a Banach space which is also a ring, such that axioms (3) and (4) above hold.

Example 8. *Given Banach spaces X and Y, the space $\mathcal{B}(X, Y)$ consisting of all bounded linear operators from X to Y is clearly a Banach algebra, where multiplication is the ordinary composition of operators.*

Example 9. *If X is any Hausdorff space, then the space $C(X)$ consisting of all bounded continuous functions from X to $\mathbb{C}$ forms a CB algebra with a unit, where multiplication is defined pointwise, and the norm is the sup norm. The unit of this space is the function which is identically equal to 1. When I get to the Gelfand representation theorem, an important algebra will be $C(\mathfrak{M})$, where the space $\mathfrak{M}$ is a compact Hausdorff space.*

Definition 10. *Let A be a Banach algebra with a unit and let $x \in A$. x is said to be invertible if $\exists x^{-1} \in A$ such that $xx^{-1} = x^{-1}x = e$. I will use, without proof, the fact that the set of invertible elements in A forms an open set of A.*

3 Homomorphisms and Ideals

Definition 11. *Let A be a CB algebra. A subspace J of A is said to be an* ideal *of A if $\forall x \in A$ and $\forall j \in J$, $xj \in J$. (More generally, if A is not commutative, we require both $xj \in J$ and $jx \in J$).*

Definition 12. *Let A and B be CB algebras. An linear operator $\phi : A \to B$ is said to be a* homomorphism *if $\forall x, y \in A, \phi(xy) = \phi(x)\phi(y)$.*

Note that if $x \in \ker(\phi)$ and if $y \in A$, then $\phi(xy) = \phi(x)\phi(y) = 0 \cdot \phi(y) = 0$. Hence $\ker(\phi)$ is an ideal of A.

Definition 13. *Let A be a Banach algebra and J an ideal of A, then*

- *J is said to be* proper *if $J \neq A$*
- *If J is proper, then J is said to be* maximal *if for any proper ideal M, $J \subset M \Rightarrow M = J$.*

Lemma 14. *Let A be a Banach algebra with a unit and let J be a proper ideal of A. Then J contains no invertible elements and is not dense in A. Moreover, its closure $\bar{J}$ is also a proper ideal of A.*

Proof. Suppose $x \in J$ and $\exists x^{-1} \in A$ such that $xx^{-1} = e$. But J is an ideal, hence $e \in J$ and so $\forall m \in A$, $m = em \in J$. In which case J is not proper. For the second part, we use the fact that the invertible elements of A form an open set of A, together with the first part. For the last part, observe that for any $x \in A$, the mapping $y \mapsto x \cdot y$ is continuous on A, and that for any $j \in \bar{J}$, we may write $j = \lim_{n\to\infty} j_n$, where $\forall n, j_n \in J$. Hence for any $j \in \bar{J}$ and any $x \in A$, $x \cdot j = x \cdot \lim_{n\to\infty} j_n = \lim_{n\to\infty} x \cdot j_n$ and the last quantity is in $\bar{J}$ because $\bar{J}$ is closed. Consequently, $\bar{J}$ is an ideal of A, and must be proper as J isn't dense in A. □

Corollary 15. *Every maximal ideal is closed and every proper ideal is contained in a maximal ideal.*

Proof. Let J be a maximal ideal of A. Then by the previous theorem, we know that $\bar{J}$ is a proper ideal of A which contains J, but by maximality, $J = \bar{J}$. The second part is proved by a standard Zorn's lemma trick. □

Of particular importance to the Gelfand representation theorem are the homomorphisms into $\mathbb{C}$, ie. the multiplicative linear functionals. First note the following result about linear functionals:

Lemma 16. *Let ϕ be a linear functional on a normed linear space X, then:*

1. *If ϕ is nonzero, then $\ker(\phi)$ is a maximal subspace of X (ie. a maximal element of the set of proper subspaces of X).*
2. *ϕ is continuous iff $\ker(\phi)$ is closed.*
3. *If ϕ is a nonzero homomorphism, then $\phi(e) = 1$.*

Proof. Suppose ϕ is nonzero. This immediately gives $\ker(\phi) \neq X$, so $\ker(\phi)$ is certainly a proper subpace of X. Also $\mathrm{Im}(\phi)$ is nonzero subspace of $\mathbb{C}$, so we have $\mathrm{Im}(\phi) = \mathbb{C}$. By the first isomorphism theorem, $X/\ker(\phi) \cong \mathbb{C}$. Now suppose S is a subpace of X which contains $\ker(\phi)$, but there exists a 1-1 correspondence between the subspaces of the quotient space $X/\ker(\phi)$ and the subspaces of X which contain $\ker(\phi)$, so if $S \neq \ker(\phi)$, we must have $X/S = \{0\}$, in which case $S = X$. Therefore $\ker(\phi)$ must be maximal, as required.

Suppose ϕ is continuous. Let y_n be a convergent sequence of elements of $\ker(\phi)$, say $y_n \to y$. But by continuity, $0 \equiv \phi(y_n) \to \phi(y)$, so $\phi(y) = 0$, so $y \in \ker(\phi)$. Hence $\ker(\phi)$ is closed.

Suppose $\ker(\phi)$ is closed. I will prove that ϕ is continuous at 0, and hence continuous on X. If $\phi \equiv 0$, then there is nothing to prove, otherwise, $\mathrm{Im}(\phi) = \mathbb{C}$, hence $\forall \varepsilon \in \mathbb{R}, \exists x_\varepsilon$ such that $\phi(x_\varepsilon) = \varepsilon$. Define $K_\varepsilon = \{x : \phi(x) = \varepsilon\}$. Then $K_\varepsilon = \{x : \phi(x - x_\varepsilon) = 0\}$, and so $K_\varepsilon = K_0 + x_\varepsilon$. But, by hypothesis, K_0 is closed, hence K_ε is closed. Consequently, for each $\varepsilon \in \mathbb{R}$, $\{x : \phi(x) \neq \varepsilon\}$ is open, and

$$U_\varepsilon := \{x : |\phi(x)| < \varepsilon\} = \bigcup_{|\delta| \geq \varepsilon} \{x : \phi(x) \neq \delta\}$$

is also open. Hence ϕ is continous at 0.

For the last part, observe that $\phi(e) \neq 0$ (otherwise $\forall x \in X$, $\phi(x) = \phi(ex) = \phi(e)\phi(x) = 0 \cdot \phi(x) = 0$, making ϕ identically zero), hence $\phi(e) = \phi(ee) = \phi(e)\phi(e)$, and making use of the fact that $\mathbb{C}$ is a field, we have $\phi(e) = e_{\mathbb{C}} = 1$. □

Corollary 17. *Let A be a Banach space with a unit, and $\phi : A \to \mathbb{C}$ a homomorphism, then ϕ is continuous and if $\phi \neq 0$, then $\|\phi\| = 1$.*

Proof. $\ker(\phi)$ is a maximal subspace, hence a maximal ideal, hence closed. Thus ϕ is continuous. It is certainly the case that $\|\phi\| \geq 1$ as $\phi(e) = 1$. Suppose $\|\phi\| > 1$ then $\exists x \in A$ such that $\|x\| \leq 1$ and $\phi(x) > 1$ but for all n, $\|x^n\| \leq \|x\|^n \leq 1$, though $\phi(x^n) = \phi(x)^n \to \infty$, but ϕ is bounded, a contradiction. □

3.1 A Spectral Result

An essential result I'll need in showing that the Gelfand representation is norm-decreasing is the following:

Theorem 18. *Let A be a Banach algebra with a unit and $\phi : A \to \mathbb{C}$ is a nonzero homomorphism, then $\forall x \in A$, $\phi(x) \in \sigma(x)$, where, by definition, the spectrum $\sigma(x)$ is given by $\sigma(x) = \{\lambda \in \mathbb{C} : \lambda e - x \text{ is not invertible}\}$.*

Proof. We have to show that $\phi(x)e - x$ is not invertible. Now $\phi(e) - x \in \ker(\phi)$, but $\ker(\phi)$ is a maximal subspace, hence contains no invertible elements. □

We introduce the notion of the spectrum of an element of a unital Banach algebra because it is related to the spectra of linear operators. Indeed, for each $x \in A$, define a mapping $L_x : A \to A$ by $L_x(y) = xy$. It is easily seen that L_x is a bounded linear operator on A, and that the mapping $x \longmapsto L_x$ is an isometric isomorphism of A onto a closed subspace of $\mathcal{B}(A)$. In fact, the mapping is also multiplicative, and also x is invertible if and only if L_x is invertible.

By the last statement, we have that $\sigma(x) = \sigma(L_x)$.

We define the spectral radius $r_\sigma(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$.

Lemma 19. *For any $x \in A$, $r_\sigma(x) = \lim_{n\to\infty} \|x^n\|^{1/n}$ and $r_\sigma(x) \leq \|x\|$.*

Proof. Using, without proof, the result that for a linear operator L, $r_\sigma(L) = \lim_{n\to\infty} \|L^n\|^{1/n}$, we have that

$$r_\sigma(x) = r_\sigma(L_x) = \lim_{n\to\infty} \|(L_x)^n\|^{1/n} = \lim_{n\to\infty} \|L_{x^n}\|^{1/n} = \lim_{n\to\infty} \|x^n\|^{1/n}$$

But, because A is a Banach algebra, $\|x^n\| \le \|x\|^n$, so $r_\sigma(x) \le \|x\|$. □

4 The Gelfand Representation Theorem

The Gelfand representation theorem is an omnibus theorem concerning a mapping (the Gelfand representation) between a CB algebra A with a unit and an associated algebra $\hat{A}$ consisting of continuous functions on the so-called carrier space $\mathfrak{M}$ of A. Before we can state the theorem, some definitions are in order.

Definition 20. *The carrier space $\mathfrak{M}$ of A is the space of all nonzero multiplicative linear functionals (ie. homomorphisms from A into the space of scalars), endowed with the subspace topology which it inherits from the dual space A^*, equipped with the weak-star topology.*

Note that $\mathfrak{M}$ truly is a subset of A^*, because every element of $\mathfrak{M}$ is a bounded linear functional, by Corollary 17.

Definition 21. *For each $x \in A$, the* Gelfand transform *of x is the function $\hat{x} : \mathfrak{M} \to \mathbb{C}$ defined by $\hat{x}(\phi) = \phi(x)$ for all $\phi \in \mathfrak{M}$.*

By Remark 3, each function $\hat{x}$ is continuous with respect to the weak-star topology and again using Corollary 17, we have $|\hat{x}(\phi)| = |\phi(x)| \le \|\phi\| \|x\| = \|x\|$, that is, each function $\hat{x}$ is also bounded, and hence each $\hat{x} \in C(\mathfrak{M})$, the space of bounded continuous functions from $\mathfrak{M}$ to $\mathbb{C}$.

Theorem 22 (Gelfand Representation Theorem)**.** *Let A be a CB algebra with a unit. Then*

1. *its carrier space $\mathfrak{M}$ is a compact Hausdorff space.*
2. *$\forall x \in A$, $\hat{x}$ is a continuous function on $\mathfrak{M}$ and the space $\hat{A} := \{\hat{x} : x \in A\}$ is a closed subalgebra of the algebra $C(\mathfrak{M})$ of all continuous functions on $\mathfrak{M}$.*
3. *The Gelfand representation $x \mapsto \hat{x}$ is a norm-decreasing homomorphism onto $\hat{A}$.*
4. *$\forall \phi \in \mathfrak{M}$, $\hat{e}(\phi) = 1$.*
5. *Each constant function is contained in $\hat{A}$ and $\hat{A}$ separates the points of $\mathfrak{M}$. (that is, $\forall \phi_1, \phi_2 \in \mathfrak{M}$ with $\phi_1 \ne \phi_2$, $\exists \hat{x} \in \hat{A}, \hat{x}(\phi_1) \ne \hat{x}(\phi_2)$)*
6. *$\hat{x}$ is invertible in $C(\mathfrak{M})$ iff x is invertible in A.*

7. $\|\hat{x}\|_\infty = \lim_{n\to\infty} \|x^n\|^{1/n}$.

8. *$\hat{A}$ is isomorphic to A iff A is semisimple (that is, the intersection of all maximal ideals of A is $\{0\}$).*

Proof. We present the proof in a number of steps. First the fact that $\mathfrak{M}$ is compact and Hausdorff.

By Theorem 5, the weak-star topology is Hausdorff, hence any (topological) subspace is also Hausdorff.

Now from Theorem 6, the unit ball $B(A^*)$ is weak-star compact, but recall that each element of $\mathfrak{M}$ has unit norm (as a subset of A^* with the operator topology), so $\mathfrak{M} \subset B(A^*)$, so we are left to show that $\mathfrak{M}$ is closed (as a closed subset of a compact set is compact). It suffices to show that if $z \in \overline{\mathfrak{M}}$, then z is a nonzero homomorphism.Fix $x, y \in A$ and $\varepsilon > 0$. Define $U_{xy\varepsilon}$ by

$$U_{xy\varepsilon} = \{u \in A^* : |(z-u)|(x) < \varepsilon, |(z-u)|(y) < \varepsilon, |(z-u)|(xy) < \varepsilon\}$$

$U_{xy\varepsilon}$ is seen to be a weak-star neighbourhood of z, from which we deduce $\exists \phi \in \mathfrak{M} \cap U_{xy\varepsilon}$. Therefore, as ϕ is multiplicative,

$$z(xy) - z(x)z(y) = [z(xy) - \phi(xy)] + \phi(x)[\phi(y) - z(y)] + [\phi(x) - z(x)]\, z(y)$$

so

$$\begin{aligned} |z(xy) - z(x)z(y)| &\leq \varepsilon + |\phi(x)|\,\varepsilon + \varepsilon\,|z(y)| \\ &< \varepsilon(1 + \|\phi\|\,\|x\| + \|z\|\,\|y\|) \\ &< \varepsilon(1 + \|x\| + \|y\|) \end{aligned}$$

consequently, $z(xy) = z(x)z(y)$. Also, by considering the neighbourhood $V_\varepsilon := \{u \in A^* : |(z-u)|(e) < \varepsilon\}$, and an element $\xi \in \mathfrak{M} \cap V_\varepsilon$,

$$\begin{aligned} z(e) - 1 &= [z(e) - \xi(e)] + [\xi(e) - 1] \\ &= z(e) - \xi(e) \end{aligned}$$

so $|z(e) - 1| < \varepsilon$. and, $z(e) = 1$, implying that z is nonzero. So we've shown that $\mathfrak{M}$ is compact and Hausdorff.

Observe that $\forall x, y \in A$ and $\forall \phi \in \mathfrak{M}$,

$$\widehat{xy}(\phi) = \phi(xy) = \phi(x)\phi(y) = \hat{x}(\phi)\,\hat{y}(\phi)$$

so the Gelfand representation is multiplicative, and $\forall \alpha, \beta \in \mathbb{C}$,

$$\widehat{\alpha x + \beta} y(\phi) = \phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y) = \alpha\hat{x}(\phi) + \beta\hat{y}(\phi)$$

so is is also linear. In consequence, its image $\hat{A}$ is a subalgebra of $C(\mathfrak{M})$.

To see it is norm-decreasing, note that

$$\|\hat{x}\| := \sup_{\phi \in \mathfrak{M}} |\hat{x}(\phi)| = \sup_{\phi \in \mathfrak{M}} |\phi(x)| \leq \|x\|$$

for the last inequality, use results 18 and 19. So we've shown the second and third parts

The fourth part is easy because $\hat{e}(\phi) = \phi(e) = 1$, as ϕ is assumed to be a nonzero homomorphism.

For the fifth part, let $\lambda \in \mathbb{C}$. Then $\forall \phi \in \mathfrak{M}$, $\widehat{\lambda e}(\phi) = \phi(\lambda e) = \lambda$, so each constant function is contained in $\hat{A}$. Moreover, if $\forall x \in A$, $\hat{x}(\phi_1) = \hat{x}(\phi_2)$ then $\forall x \in A$, $\phi_1(x) = \phi_2(x)$, hence $\phi_1 = \phi_2$, so $\hat{A}$ does indeed separate the points of $\mathfrak{M}$.

Note that by the Stone-Weierstrass theorem, and from the first, second and fifth parts of the theorem, $\hat{A} = C(\mathfrak{M})$. (This is precisely the conclusion of the Stone-Weierstrass theorem).

The remainder of the proof is left as an exercise for the reader. □

The importance of the Gelfand representation theorem is that it gives a criterion for determining whether or not a given unital CB algebra is isomorphic to an algebra of bounded continuous functions on a compact Hausdorff space.

References

[1] B. Bollobás, *Linear Analysis, an introductory course*, Cambridge University Press, Cambridge, 1990.

[2] A. E. Taylor & D. C. Lay, *Introduction to Functional Analysis*,2nd Ed. John Wiley & Sons, New York, 1980.