1 Topological Linear Spaces

Though Banach algebras are themselves normed linear spaces, the Gelfand representation theorem involves a linear space which is not endowed with a norm, so I will first make some comments concerning topological linear spaces.

Definition 1. Given a topological space \((X, \tau)\), a collection \(\sigma \subseteq \tau\) is said to be a sub-basis of \(\tau\) if every element of \(\tau\) is a union of finite intersections of elements of \(\sigma\).

Moreover, if we have an arbitrary collection \(\sigma\) of subsets of \(X\), then the collection of all unions of finite intersections of elements of \(\sigma\), together with \(\emptyset\) and \(X\), is easily seen to form a topology on \(X\) (and, of course, \(\sigma\) is a sub-basis for this topology).

Definition 2. Let \(X\) be any set, let \(\Gamma\) be an arbitrary index set and for each \(\gamma \in \Gamma\), let \(f_\gamma\) be a mapping from \(X\) to a topological space \((X_\gamma, \tau_\gamma)\). Let \(\mathcal{F} := \{f_\gamma : \gamma \in \Gamma\}\). Then the weak topology generated by \(\mathcal{F}\) is the topology generated by the sub-basis

\[\sigma = \{f_\gamma^{-1}(U_\gamma) : U_\gamma \in \tau_\gamma (\gamma \in \Gamma)\}\]

this topology is denoted by \(\sigma(X, \mathcal{F})\).

Remark 3. The importance of the weak topology is that it is the weakest topology for which each of the functions \(f_\gamma\) is continuous from \((X, \sigma(X, \mathcal{F}))\) to \((X_\gamma, \tau_\gamma)\).

Definition 4. Let \(X\) be a normed linear space, \(X^*\) its dual, \(X^{**}\) its second dual and for each \(x \in X\) let \(\hat{x}\) denote the corresponding element of \(X^{**}\) as described in lectures, then the weak-star topology \(\sigma(X^*, X)\) on \(X^*\) is the topology generated by the elements \(\hat{x}\).

Without proof, I will use the following results for a normed linear space \(X\):

Theorem 5. The weak-star topology \(\sigma(X^*, X)\) is Hausdorff.

Theorem 6 (Alaoglu). The unit ball \(B(X^*) := \{x \in X : \|x\| \leq 1\}\) is compact in the weak-star topology.
2 Banach Algebras

Banach algebras are a kind of Banach space which generalise the spaces of bounded linear operators. Indeed, we have the following definition:

**Definition 7.** Let $A$ be a complex Banach space. $A$ is said to be a Banach algebra if there is a multiplication defined on $A$ such that $\forall \lambda \in \mathbb{C}$ and $\forall x, y, z \in A$,

1. $x(yz) = (xy)z$;
2. $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;
3. $\lambda(xy) = (\lambda x)y = x(\lambda y)$;
4. $\|xy\| \leq \|x\| \|y\|$.

Moreover, the Banach algebra is said to have a unit if $\exists e \in A$ such that $\forall x \in A$, $ex = xe = x$ and $\|e\| = 1$. The algebra is said to be commutative if for any $x, y \in A$, $xy = yx$. I shall refer to commutative Banach algebras as CB algebras.

Note an equivalent definition of a Banach algebra is that it is a Banach space which is also a ring, such that axioms (3) and (4) above hold.

**Example 8.** Given Banach spaces $X$ and $Y$, the space $B(X, Y)$ consisting of all bounded linear operators from $X$ to $Y$ is clearly a Banach algebra, where multiplication is the ordinary composition of operators.

**Example 9.** If $X$ is any Hausdorff space, then the space $C(X)$ consisting of all bounded continuous functions from $X$ to $\mathbb{C}$ forms a CB algebra with a unit, where multiplication is defined pointwise, and the norm is the sup norm. The unit of this space is the function which is identically equal to 1. When I get to the Gelfand representation theorem, an important algebra will be $C(\mathcal{M})$, where the space $\mathcal{M}$ is a compact Hausdorff space.

**Definition 10.** Let $A$ be a Banach algebra with a unit and let $x \in A$. $x$ is said to be invertible if $\exists x^{-1} \in A$ such that $xx^{-1} = x^{-1}x = e$. I will use, without proof, the fact that the set of invertible elements in $A$ forms an open set of $A$.

3 Homomorphisms and Ideals

**Definition 11.** Let $A$ be a CB algebra. A subspace $J$ of $A$ is said to be an ideal of $A$ if $\forall x \in A$ and $\forall j \in J$, $xj \in J$. (More generally, if $A$ is not commutative, we require both $xj \in J$ and $jx \in J$).

**Definition 12.** Let $A$ and $B$ be CB algebras. An linear operator $\phi : A \to B$ is said to be a homomorphism if $\forall x, y \in A$, $\phi(xy) = \phi(x)\phi(y)$.

Note that if $x \in \ker(\phi)$ and if $y \in A$, then $\phi(xy) = \phi(x)\phi(y) = 0 \cdot \phi(y) = 0$. Hence $\ker(\phi)$ is an ideal of $A$. 


Definition 13. Let $A$ be a Banach algebra and $J$ an ideal of $A$, then

- $J$ is said to be proper if $J \neq A$
- If $J$ is proper, then $J$ is said to be maximal if for any proper ideal $M$, $J \subset M \Rightarrow M = J$.

Lemma 14. Let $A$ be a Banach algebra with a unit and let $J$ be a proper ideal of $A$. Then $J$ contains no invertible elements and is not dense in $A$. Moreover, its closure $\overline{J}$ is also a proper ideal of $A$.

Proof. Suppose $x \in J$ and $\exists x^{-1} \in A$ such that $xx^{-1} = e$. But $J$ is an ideal, hence $e \in J$ and so $\forall m \in A$, $m = em \in J$. In which case $J$ is not proper. For the second part, we use the fact that the invertible elements of $A$ form an open set of $A$, together with the first part. For the last part, observe that for any $x \in A$, the mapping $y \mapsto x \cdot y$ is continuous on $A$, and that for any $j \in J$, we may write $j = \lim_{n \to \infty} j_n$, where $\forall n, j_n \in J$. Hence for any $j \in J$ and any $x \in A$, $x \cdot j = \lim_{n \to \infty} x \cdot j_n$ and the last quantity is in $J$ because $J$ is closed. Consequently, $J$ is an ideal of $A$, and must be proper as $J$ isn’t dense in $A$. 

Corollary 15. Every maximal ideal is closed and every proper ideal is contained in a maximal ideal.

Proof. Let $J$ be a maximal ideal of $A$. Then by the previous theorem, we know that $J$ is a proper ideal of $A$ which contains $J$, but by maximality, $J = \overline{J}$. The second part is proved by a standard Zorn’s lemma trick. 

Of particular importance to the Gelfand representation theorem are the homomorphisms into $\mathbb{C}$, i.e. the multiplicative linear functionals. First note the following result about linear functionals:

Lemma 16. Let $\phi$ be a linear functional on a normed linear space $X$, then:

1. If $\phi$ is nonzero, then $\ker(\phi)$ is a maximal subspace of $X$ (i.e. a maximal element of the set of proper subspaces of $X$).

2. $\phi$ is continuous iff $\ker(\phi)$ is closed.

3. If $\phi$ is a nonzero homomorphism, then $\phi(e) = 1$.

Proof. Suppose $\phi$ is nonzero. This immediately gives $\ker(\phi) \neq X$, so $\ker(\phi)$ is certainly a proper subspace of $X$. Also $\text{Im}(\phi)$ is nonzero subspace of $\mathbb{C}$, so we have $\text{Im}(\phi) = \mathbb{C}$. By the first isomorphism theorem, $X/\ker(\phi) \cong \mathbb{C}$. Now suppose $S$ is a subspace of $X$ which contains $\ker(\phi)$, but there exists a 1-1 correspondence between the subspaces of the quotient space $X/\ker(\phi)$ and the subspaces of $X$ which contain $\ker(\phi)$, so if $S \neq \ker(\phi)$, we must have $X/S = \{0\}$, in which case $S = X$. Therefore $\ker(\phi)$ must be maximal, as required.
Suppose \( \phi \) is continuous. Let \( y_n \) be a convergent sequence of elements of \( \ker (\phi) \), say \( y_n \to y \). But by continuity, \( 0 \equiv \phi(y_n) \to \phi(y) \), so \( \phi(y) = 0 \), so \( y \in \ker (\phi) \). Hence \( \ker (\phi) \) is closed.

Suppose \( \ker (\phi) \) is closed. I will prove that \( \phi \) is continuous at 0, and hence continuous on \( X \). If \( \phi \equiv 0 \), then there is nothing to prove, otherwise, \( \Im(\phi) = \mathbb{C} \), hence \( \forall \varepsilon \in \mathbb{R}, \exists \varepsilon_x \) such that \( \phi(x_x) = \varepsilon \). Define \( K_\varepsilon = \{ x : \phi(x) = \varepsilon \} \). Then \( K_\varepsilon = \{ x : \phi(x - x_x) = 0 \} \), and so \( K_\varepsilon = K_0 + x_x \). But, by hypothesis, \( K_0 \) is closed, hence \( K_\varepsilon \) is closed. Consequently, for each \( \varepsilon \in \mathbb{R} \), \( \{ x : \phi(x) \neq \varepsilon \} \) is open, and

\[
U_\varepsilon := \{ x : |\phi(x)| < \varepsilon \} = \bigcup_{|\delta| \geq \varepsilon} \{ x : \phi(x) \neq \delta \}
\]

is also open. Hence \( \phi \) is continuous at 0.

For the last part, observe that \( \phi(e) \neq 0 \) (otherwise \( \forall x \in X, \phi(x) = \phi(ex) = \phi(e)\phi(x) = 0 \cdot \phi(x) = 0 \), making \( \phi \) identically zero), hence \( \phi(e) = \phi(ee) = \phi(e)\phi(e) \), and making use of the fact that \( C \) is a field, we have \( \phi(e) = e_\mathbb{C} = 1 \).

**Corollary 17.** Let \( A \) be a Banach space with a unit, and \( \phi : A \to \mathbb{C} \) a homomorphism, then \( \phi \) is continuous and if \( \phi \neq 0 \), then \( ||\phi|| = 1 \).

**Proof.** \( \ker (\phi) \) is a maximal subspace, hence a maximal ideal, hence closed. Thus \( \phi \) is continuous. It is certainly the case that \( ||\phi|| \geq 1 \) as \( \phi(e) = 1 \). Suppose \( ||\phi|| > 1 \) then \( \exists e \in A \) such that \( ||x|| \leq 1 \) and \( \phi(x) > 1 \) but for all \( n \), \( ||x^n|| \leq ||x|| \leq 1 \), though \( \phi(x^n) = \phi(x)^n \to \infty \), but \( \phi \) is bounded, a contradiction. \(\square\)

### 3.1 A Spectral Result

An essential result I’ll need in showing that the Gelfand representation is norm-decreasing is the following:

**Theorem 18.** Let \( A \) be a Banach algebra with a unit and \( \phi : A \to \mathbb{C} \) is a nonzero homomorphism, then \( \forall x \in A, \phi(x) \in \sigma (x) \), where, by definition, the spectrum \( \sigma (x) \) is given by \( \sigma (x) = \{ \lambda \in \mathbb{C} : \lambda e - x \text{ is not invertible} \} \).

**Proof.** We have to show that \( \phi(x)e - x \) is not invertible. Now \( \phi(e) - x \in \ker (\phi) \), but \( \ker (\phi) \) is a maximal subspace, hence contains no invertible elements. \(\square\)

We introduce the notion of the spectrum of an element of a unital Banach algebra because it is related to the spectra of linear operators. Indeed, for each \( x \in A \), define a mapping \( L_x : A \to A \) by \( L_x(y) = xy \). It is easily seen that \( L_x \) is a bounded linear operator on \( A \), and that the mapping \( x \mapsto L_x \) is an isometric isomorphism of \( A \) onto a closed subspace of \( B (A) \). In fact, the mapping is also multiplicative, and also \( x \) is invertible if and only if \( L_x \) is invertible.

By the last statement, we have that \( \sigma (x) = \sigma (L_x) \).

We define the spectral radius \( r_\sigma(x) = \sup \{ |\lambda| : \lambda \in \sigma (x) \} \).

**Lemma 19.** For any \( x \in A \), \( r_\sigma(x) = \lim_{n \to \infty} ||x^n||^{1/n} \) and \( r_\sigma(x) \leq ||x|| \).
Proof. Using, without proof, the result that for a linear operator \( L \), \( r_\sigma(L) = \lim_{n \to \infty} \|L^n\|^{1/n} \), we have that
\[
r_\sigma(x) = r_\sigma(Lx) = \lim_{n \to \infty} \|(Lx)^n\|^{1/n} = \lim_{n \to \infty} \|Lx^n\|^{1/n} = \lim_{n \to \infty} \|x^n\|^{1/n}
\]
But, because \( A \) is a Banach algebra, \( \|x^n\| \leq \|x\|^n \), so \( r_\sigma(x) \leq \|x\| \).

4 The Gelfand Representation Theorem

The Gelfand representation theorem is an omnibus theorem concerning a mapping (the Gelfand representation) between a CB algebra \( A \) with a unit and an associated algebra \( \hat{A} \) consisting of continuous functions on the so-called carrier space \( \mathcal{M} \) of \( A \). Before we can state the theorem, some definitions are in order.

Definition 20. The carrier space \( \mathcal{M} \) of \( A \) is the space of all nonzero multiplicative linear functionals (i.e. homomorphisms from \( A \) into the space of scalars), endowed with the subspace topology which it inherits from the dual space \( A^* \), equipped with the weak-star topology.

Note that \( \mathcal{M} \) truly is a subset of \( A^* \), because every element of \( \mathcal{M} \) is a bounded linear functional, by Corollary 17.

Definition 21. For each \( x \in A \), the Gelfand transform of \( x \) is the function \( \hat{x} : \mathcal{M} \to \mathbb{C} \) defined by \( \hat{x}(\phi) = \phi(x) \) for all \( \phi \in \mathcal{M} \).

By Remark 3, each function \( \hat{x} \) is continuous with respect to the weak-star topology and again using Corollary 17, we have \( |\hat{x}(\phi)| = |\phi(x)| \leq \|\phi\| \|x\| = \|x\| \), that is, each function \( \hat{x} \) is also bounded, and hence each \( \hat{x} \in C(\mathcal{M}) \), the space of bounded continuous functions from \( \mathcal{M} \) to \( \mathbb{C} \).

Theorem 22 (Gelfand Representation Theorem). Let \( A \) be a CB algebra with a unit. Then

1. its carrier space \( \mathcal{M} \) is a compact Hausdorff space.
2. \( \forall x \in A, \, \hat{x} \) is a continuous function on \( \mathcal{M} \) and the space \( \hat{A} := \{ \hat{x} : x \in A \} \) is a closed subalgebra of the algebra \( C(\mathcal{M}) \) of all continuous functions on \( \mathcal{M} \).
3. The Gelfand representation \( x \mapsto \hat{x} \) is a norm-decreasing homomorphism onto \( \hat{A} \).
4. \( \forall \phi \in \mathcal{M}, \, \hat{\phi} = 1 \).
5. Each constant function is contained in \( \hat{A} \) and \( \hat{A} \) separates the points of \( \mathcal{M} \) (that is, \( \forall \phi_1, \phi_2 \in \mathcal{M} \) with \( \phi_1 \neq \phi_2 \), \( \exists \hat{x} \in \hat{A}, \hat{x}(\phi_1) \neq \hat{x}(\phi_2) \))
6. \( \hat{x} \) is invertible in \( C(\mathcal{M}) \) iff \( x \) is invertible in \( A \).
7. \( \|\hat{x}\|_\infty = \lim_{n \to \infty} \|x^n\|^{1/n} \).

8. \( \hat{A} \) is isomorphic to \( A \) iff \( A \) is semisimple (that is, the intersection of all maximal ideals of \( A \) is \( \{0\} \)).

Proof. We present the proof in a number of steps. First the fact that \( \mathfrak{M} \) is compact and Hausdorff.

By Theorem 5, the weak-star topology is Hausdorff, hence any (topological) subspace is also Hausdorff.

Now from Theorem 6, the unit ball \( B(A^*) \) is weak-star compact, but recall that each element of \( \mathfrak{M} \) has unit norm (as a subset of \( A^* \) with the operator topology), so \( \mathfrak{M} \subset B(A^*) \), so we are left to show that \( \mathfrak{M} \) is closed (as a closed subset of a compact set is compact). It suffices to show that if \( z \in \mathfrak{M} \), then \( z \) is a nonzero homomorphism. Fix \( x, y \in A \) and \( \varepsilon > 0 \). Define \( U_{xy\varepsilon} \) by

\[
U_{xy\varepsilon} = \{ u \in A^* : |(z - u)(x)| < \varepsilon, |(z - u)(y)| < \varepsilon, |(z - u)(xy)| < \varepsilon \}
\]

\( U_{xy\varepsilon} \) is seen to be a weak-star neighbourhood of \( z \), from which we deduce \( \exists \phi \in \mathfrak{M} \cap U_{xy\varepsilon} \). Therefore, as \( \phi \) is multiplicative,

\[
z(xy) - z(x)z(y) = [z(xy) - \phi(xy)] + \phi(x)[\phi(y) - z(y)] + [\phi(x) - z(x)]z(y)
\]

so

\[
|z(xy) - z(x)z(y)| \leq \varepsilon + |\phi(x)| \varepsilon + \varepsilon |z(y)|
\]

\[
< \varepsilon(1 + \|\phi\| \|x\| + \|z\| \|y\|)
\]

\[
< \varepsilon(1 + \|x\| + \|y\|)
\]

consequently, \( z(xy) = z(x)z(y) \). Also, by considering the neighbourhood \( V_{\varepsilon} := \{ u \in A^* : |(z - u)(e)| < \varepsilon \} \), and an element \( \xi \in \mathfrak{M} \cap V_{\varepsilon} \),

\[
z(e) - 1 = [z(e) - \xi(e)] + [\xi(e) - 1]
= z(e) - \xi(e)
\]

so \( |z(e) - 1| < \varepsilon \). and, \( z(e) = 1 \), implying that \( z \) is nonzero. So we’ve shown that \( \mathfrak{M} \) is compact and Hausdorff.

Observe that \( \forall x, y \in A \) and \( \forall \phi \in \mathfrak{M} \),

\[
\hat{x}\hat{y}(\phi) = \phi(xy) = \phi(x)\phi(y) = \hat{x}(\phi)\hat{y}(\phi)
\]

so the Gelfand representation is multiplicative, and \( \forall \alpha, \beta \in \mathbb{C} \),

\[
\alpha\hat{x} + \beta\hat{y}(\phi) = \phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y) = \alpha\hat{x}(\phi) + \beta\hat{y}(\phi)
\]

so is is also linear. In consequence, its image \( \hat{A} \) is a subalgebra of \( C(\mathfrak{M}) \).

To see it is norm-decreasing, note that

\[
\|\hat{x}\| := \sup_{\phi \in \mathfrak{M}} |\hat{x}(\phi)| = \sup_{\phi \in \mathfrak{M}} |\phi(x)| \leq \|x\|
\]
for the last inequality, use results 18 and 19. So we’ve shown the second and third parts.

The fourth part is easy because \( \hat{e}(\phi) = \phi(e) = 1 \), as \( \phi \) is assumed to be a nonzero homomorphism.

For the fifth part, let \( \lambda \in \mathbb{C} \). Then \( \forall \phi \in \mathfrak{M}, \hat{\lambda e}(\phi) = \phi(\lambda e) = \lambda \), so each constant function is contained in \( \hat{A} \). Moreover, if \( \forall x \in A, \hat{x}(\phi_1) = \hat{x}(\phi_2) \) then \( \forall x \in A, \phi_1(x) = \phi_2(x) \), hence \( \phi_1 = \phi_2 \), so \( \hat{A} \) does indeed separate the points of \( \mathfrak{M} \).

Note that by the Stone-Weierstrass theorem, and from the first, second and fifth parts of the theorem, \( \hat{A} = C(\mathfrak{M}) \). (This is precisely the conclusion of the Stone-Weierstrass theorem).

The remainder of the proof is left as an exercise for the reader. \( \square \)

The importance of the Gelfand representation theorem is that it gives a criterion for determining whether or not a given unital CB algebra is isomorphic to an algebra of bounded continuous functions on a compact Hausdorff space.

References
